§4.1 Geometry of Simplicial Complexes

Definition Let V be a vector space over $\mathbb{R}^1$ and let C be a subset of V. C is *convex* if

$$c_1\, c_2 \in C \implies tc_1 + (1-t)c_2 \in C \quad \text{for all } t \in I = [0,\, 1].$$

In $\mathbb{R}^2$, $\{v_0,\, v_1,\, v_2\}$ is c-independent if and only if v_0, v_1 and v_2 are not collinear.

Definition A set $\{v_0,\, v_1, \ldots,\, v_k\}$ of vectors in a vector space V is *convex-independent* or *c-independent* if the set $\{v_1 -_0,\, v_2 - v_0, \ldots,\, v_k - v_0\}$ is linearly independent. since

$$0 = \sum_{j \neq \ell} a_j\,(v_j - v_\ell) = \sum_{j \neq \ell} a_j\,(v_j - v_0) - \left(\sum_{j \neq \ell} a_j\right)(v_\ell - v_0) \implies a_j = 0 \text{ for all } j \neq \ell,$$

this definition does not depend on which vector is called v_0.

Example In $\mathbb{R}^2$, $\{v_0,\, v_1,\, v_2\}$ is c-independent if and only if v_0, v_1 and v_2 are not collinear.

Theorem Suppose $\{v_0,\, v_1, \ldots,\, v_k\}$ is a c-independent set. Let C be the convex set generated by $\{v_0,\, v_1, \ldots,\, v_k\}$. Then

$$C = \left\{ v = \sum_{i=0}^{k} a_i v_i \mid a_i \geq 0 \text{ for all } i \text{ and } \sum_{i=0}^{k} a_i = 1 \right\} = [v_0,\, v_1, \ldots,\, v_k]$$

and it is called a *k-simplex* (or a *closed simplex of dimension k*). Furthermore, each $v \in C$ is uniquely expressible in the form $v = \sum_{i=0}^{k} a_i v_i$ with $a_i \geq 0$ and $\sum_{i=0}^{k} a_i = 1$, where the coefficients a_i, are called the *barycentric coordinates of v*.

Example For $\{v_0,\, v_1\}$ vectors in $\mathbb{R}^1$, the simplex $[v_0,\, v_1]$ is the closed interval $[v_0,\, v_1]$. For $\{v_0,\, v_1,\, v_2\}$ in $\mathbb{R}^2$, $[v_0,\, v_1,\, v_2]$ is the triangle with vertices v_0, v_1 and v_2. The centroid of this triangle is the point with barycentric coordinates $(1/3,\, 1/3,\, 1/3)$. For $V = \mathbb{R}^n$, the simplex $[v_0,\, v_1, \ldots,\, v_k]$ is a compact metric space (it is closed and bounded) in the relative topology. In fact, using barycentric coordinates, it is not difficult to see that $[v_0,\, v_1, \ldots,\, v_k]$ is homeomorphic to a product of k unit intervals. However, this homeomorphism is not an isometry.

Definition Let $\{v_0,\, v_1, \ldots,\, v_k\}$ be a c-independent set. The set

$$\left\{ v \in [v_0,\, v_1, \ldots,\, v_k] \mid a_i > 0 \text{ for all } i \text{ and } \sum_{i=0}^{k} a_i = 1 \right\} = (v_0,\, v_1, \ldots,\, v_k)$$

is called an *open simplex.* We shall also denote an open simplex by (s) and the corresponding closed simplex by $[s]$.

Let $[s] = [v_0,\, v_1, \ldots,\, v_k]$ be a closed simplex. The *vertices* of $[s]$ are the points $v_0,\, v_1, \ldots,\, v_k$. The *closed faces* of $[s]$ are the closed simplices $[v_{j_0},\, v_{j_1}, \ldots,\, v_{j_h}]$, where $\{j_0,\, j_1, \ldots, j_h\}$ is a nonempty subset of $\{0,\, 1, \ldots, k\}$. The *open faces* of the simplex $[s]$ are the open simplices $(v_{j_0},\, v_{j_1}, \ldots,\, v_{j_h})$.

Remarks

(1) A vertex is a 0-dimensional closed face. It is also an open face.

(2) An open simplex (s) is an open set in the closed simplex $[s]$. Its closure is $[s]$.

(3) The closed simplex $[s]$ is the union of its open faces.

(4) Distinct open faces of a simplex are disjoint.

(5) The open simplex (s) is the interior of the closed simplex $[s]$; that is, it is the closed simplex minus its proper open faces (faces $\neq (s)$).

Definition A *simplicial complex* K (Euclidean) is a finite set of open simplices in some $\mathbb{R}^n$ such that

(1) if $(s) \in K$, then all open faces of $[s] \in K$;

(2) if (s_1), $(s_2) \in K$, and $(s_1) \cap (s_2) \neq \emptyset$, then $(s_1) = (s_2)$.
The *dimension* of K is the maximum dimension of the simplices of K.

Remarks If K is a simplicial complex, let $[K]$ denote the point set union of the open simplices of K. Then $[K]$ is compact, and $[K] = \bigcup_{(s) \in K} (s) = \bigcup_{(s) \in K} [s]$.

If $[s]$ is a closed simplex, the collection of its open faces is a *simplicial complex* which we denote by s.

Examples Figure 4.2 shows examples of simplicial complexes. Those shown in Figure 4.3 are not simplicial complexes. By adding simplices, however, the point sets in Figure 4.3 can be made into complexes (Figure 4.4). Note that a complex is more than just a point set. It is a set with additional structure. It is possible to have two different complexes with the same point set, as in Figure 4.5.

Figure 4.2

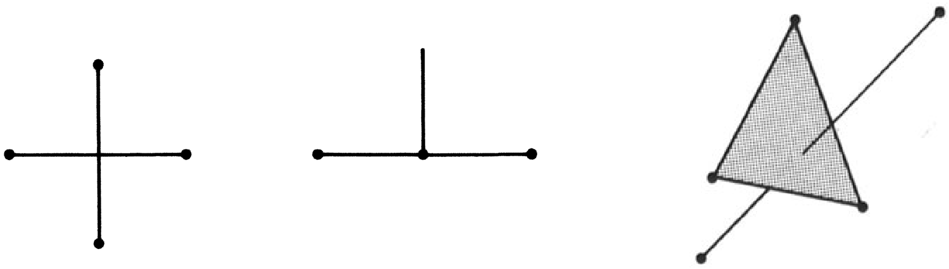


Figure 4.3

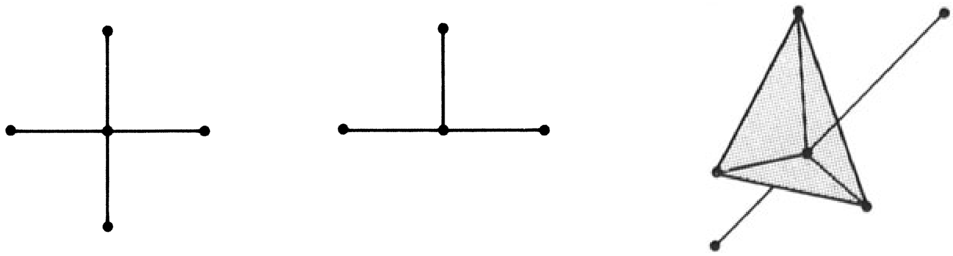


Figure 4.4

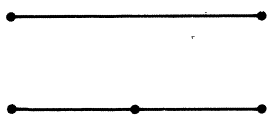


Figure 4.5

Definition A *subcomplex* of a simplicial complex K is a simplicial complex L such that $(s) \in L$ implies $(s) \in K$.

Remark For each $(s) \in K$, the simplicial complex s is a subcomplex of K.

Definition Let K be a complex. Let r be an integer less than or equal to $\dim K$. The *r-skeleton K^r* of K is the collection $K^r = \{(s) \in K \mid \dim s \leq r\}$.

Remark The r-skeleton K^r is a subcomplex of K.

§4.3 Simplicial Approximation Theorem

Definition Let K and L be simplicial complexes. A map $\varphi : [K] \to [L]$ is a *simplicial map* if

(1) for each vertex v of K, $\varphi(v)$ is a vertex of L,

(2) for each simplex $(v_0, v_1, \ldots, v_k) \in K$, the vertices $\varphi(v_0), \varphi(v_1), \ldots, \varphi(v_k)$ all lie in some closed simplex (of dimension $\leq k$) in L, and

(3) for each $(s) = (v_0, v_1, \ldots, v_k) \in K$, and $p = \sum_{i=0}^{k} a_i v_i \in (s)$, the image of p is given by

$$\varphi(p) = \sum_{i=0}^{k} a_i \varphi(v_i).$$

Examples In Figure 4.12, projection is a simplicial map. However, in Figure 4.13, projection is not a simplicial map, even though conditions (1) and (3) are satisfied.

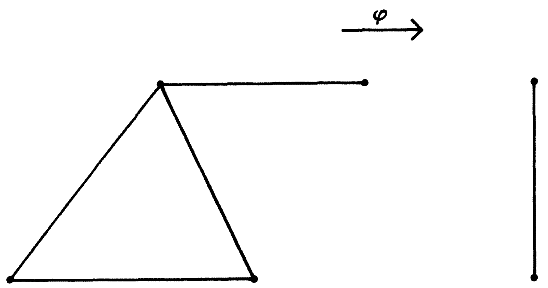


Figure 4.12

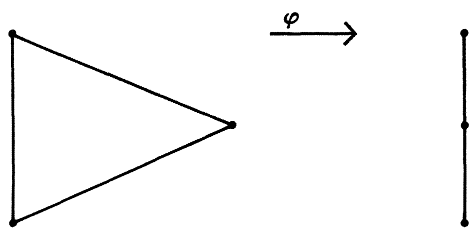


Figure 4.13

Definition Let K and L be simplicial complexes. Let $f : [K] \to [L]$ be continuous. A simplicial map $\varphi : K \to L$ is a *simplicial approximation* to f if $f(\mathrm{St}(v)) \subset \mathrm{St}(\varphi(v))$ for each vertex v of K.

Theorem 1 Let K be a simplicial complex. For v vertex of K, $\mathrm{St}(v)$ is an open set in $[K]$ containing v, and v is the only vertex of K which lies in $\mathrm{St}(v)$. The collection $\{\mathrm{St}(v)\}_{v \in K^0}$ is an open covering of $[K]$.

Theorem 2 Suppose $\varphi : K \to L$ is a simplicial approximation to $f : [K] \to [L]$. Then, for any $p \in [K]$, $f(p)$ and $\varphi(p)$ lie in a common closed simplex of $[L]$.

Theorem 3 Let φ be a simplicial approximation to $f : [K] \to [L]$. Let K_1 be a subcomplex of K, and suppose that the restriction off to $[K_1]$ is a simplicial map. Then there exists a homotopy between f and φ which is stationary on $[K_1]$.

§6.1 Simplicial Homology

We have defined the De Rham cohomology groups $H^\ell(X, d)$ for a smooth manifold X. These groups came from a sequence of maps

$$C^\infty(X, \Lambda^{\ell-1}(X)) \xrightarrow{d} C^\infty(X, \Lambda^\ell(X)) \xrightarrow{d} C^\infty(X, \Lambda^{\ell+1}(X))$$

and $H^\ell(X, d) = \operatorname{Ker} d/\operatorname{Im} d$. We saw that the dimension of $H^0(X, d)$ measured the number of connected components of X, and we saw, at least for the circle $X = S^1$, that the dimension of $H^1(X, d)$ measured the number of "holes" in X. We shall now develop similar groups for simplicial complexes. We shall study a sequence of maps

$$C_{\ell-1} \xleftarrow{\partial} C_\ell \xleftarrow{\partial} C_{\ell+1}$$

where each C_k is an abelian group and where $\partial^2 = 0$. Then homology groups H_ℓ will be defined by $H_\ell = Z_\ell/B_\ell$, where $Z_\ell = \operatorname{Ker} \partial : C_\ell \to C_{\ell-1}$ and $B_\ell = \operatorname{Im} \partial : C_{\ell+1} \to C_\ell$. An element of Z_ℓ will geometrically be a "chain" of ℓ-simplices without boundary. An element of B_ℓ, will geometrically be a boundary of a chain of $(\ell + 1)$-simplices. The boundary of a 1-simplex (v_0, v_1) will be the sum of the 0-simplices v_0 and v_1 with appropriate signs attached. Similarly, the boundary of a 2-simplex (v_0, v_1, v_2) will be an appropriate linear combination of its edges (v_0, v_1), (v_1, v_2) and (v_2, v_0).

Definition Let s be an ℓ-simplex, with vertices $v_0, v_1, \ldots, v_\ell$. Two orderings $(v_{j_0}, v_{j_1}, \ldots, v_{j_\ell})$ and $(v_{k_0}, v_{k_1}, \ldots, v_{k_\ell})$ of the vertices of s are *equivalent* if $(k_0, \ldots, k_\ell)$ is an even permutation of $(j_0, \ldots, j_\ell)$. This is clearly an equivalence relation, and for $\ell > 1$, it partitions the orderings of $v_0, v_1, \ldots, v_\ell$ into two equivalence classes. An *oriented simplex* is a simplex s together with a choice of one of these equivalence classes. If $v_0, v_1, \ldots, v_\ell$ are the vertices of s, the oriented simplex determined by the ordering $(v_0, v_1, \ldots, v_\ell)$ will be denoted by $\langle v_0, v_1, \ldots, v_\ell \rangle$.

Remark Note that an oriented ℓ-simplex has a sense of direction attached to it, an oriented 2-simplex has a sense of rotation attached to it, and so on (see Figure 6.1). In fact, each ℓ-simplex s lies in an ℓ-dimensional plane in some $\mathbb{R}^m$. Orienting s by $\langle v_0, v_1, \ldots, v_\ell \rangle$ is the same as orienting the ℓ-plane containing s by means of the ordered basis $\{v_1 - v_0, v_2 - v_0, \ldots, v_\ell - v_0\}$.

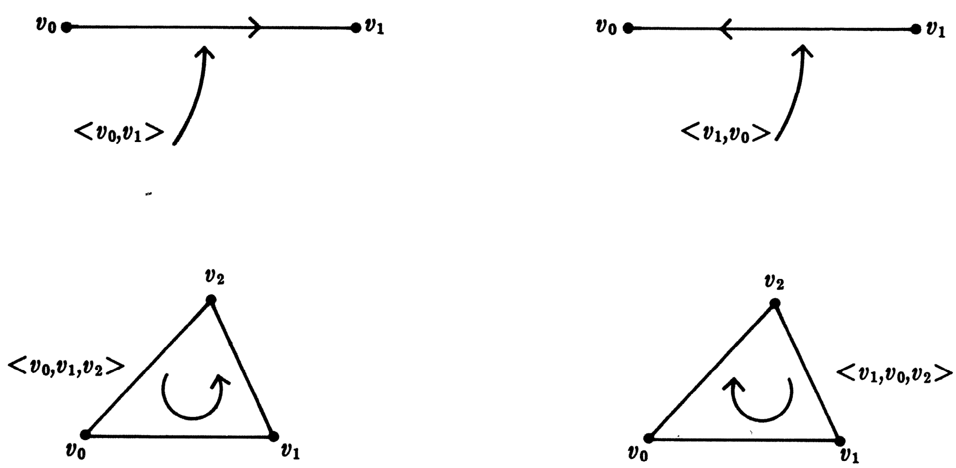


Figure 6.1

Definition Let K be a simplicial complex, and let $\mathbb{Z}$ denote the group of integers. Let $C_\ell(K, \mathbb{Z})$ denote the factor group of the free abelian group generated by all oriented simplices of K, modulo the subgroup generated by all elements of the form $\langle v_0, v_1, v_2, \ldots, v_\ell\rangle + \langle v_1, v_0, v_2, \ldots, v_\ell\rangle$. Thus $C_\ell(K, \mathbb{Z})$ is an abelian group called the *group of ℓ-chains* of K with integer coefficients. A typical element of this group is of the form

$$\sum_{s \text{ an } \ell\text{-simplex}} n_s\langle s\rangle \quad (n_s \in \mathbb{Z}),$$

where, for each ℓ-simplex s, $\langle s\rangle$ is some fixed orientation of s, and where s with the opposite orientation is identified with $-\langle s\rangle$.

Remark Given an arbitrary abelian group $\mathscr{G}$, the group $C_\ell(K, \mathscr{G})$ of ℓ-chains of K with coefficients in $\mathscr{G}$ can be defined as the set of all formal linear combinations

$$\sum_s g_s\langle s\rangle \quad (g_s \in \mathscr{G})$$

subject to the identifications $-g_s\langle v_0, v_1, \ldots, v_\ell\rangle = g_s\langle v_1, v_0, \ldots, v_\ell\rangle$. (We are writing the group operation in $\mathscr{G}$ additively.) In particular, $C_\ell(K, \mathscr{F})$ is defined for any field $\mathscr{F}$ in which case $C_\ell(K, \mathscr{F})$ is a vector space over $\mathscr{F}$ whose dimension equals the number of ℓ-simplices of K. We shall only be interested in the cases where $\mathscr{G}$ constitutes the integers $\mathbb{Z}$, the reals $\mathbb{R}$, the complexes $\mathbb{C}$, or the integers $\mathscr{I}_2$ modulo 2; that is, the group of order 2.

Definition Let $\langle s\rangle = \langle v_0, v_1, \ldots, v_{\ell+1}\rangle$ be an oriented $(\ell+1)$-simplex. The boundary $\partial\langle s\rangle$ of $\langle s\rangle$ is the ℓ-chain defined by

$$\partial\langle s\rangle = \sum_{j=0}^{\ell+1}(-1)^j\langle v_0, v_1, \ldots, \widehat{v_j}, \ldots, v_{\ell+1}\rangle$$

where $\widehat{\ }$ over a symbol means that symbol is deleted.

Remark Note that $\partial\langle s\rangle$ is well defined and that $\bigcup_{j=0}^{\ell+1}[v_0, v_1, \ldots, \widehat{v_j}, \ldots, v_{\ell+1}]$, the union ofthe faces occurring in $\partial\langle s\rangle$, is the topological boundary of $\langle s\rangle$.

Examples

(1) $\partial\langle v_0, v_1\rangle = \langle v_1\rangle - \langle v_0\rangle$.

(2) $\partial\langle v_0, v_1, v_2\rangle = \langle v_1, v_2\rangle - \langle v_0, v_2\rangle + \langle v_0, v_1\rangle = \langle v_0, v_1\rangle + \langle v_1, v_2\rangle + \langle v_2, v_0\rangle$. (see Figure 6.2)

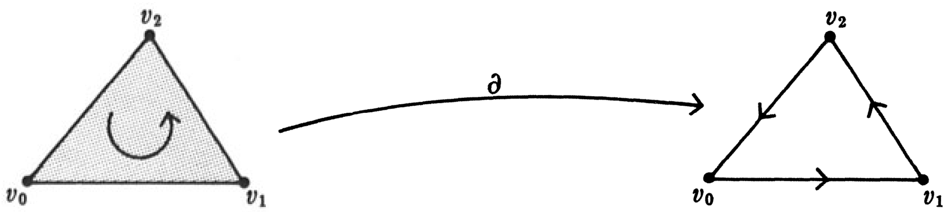


Figure 6.2

Definition Let K be a simplicial complex, and let $\mathscr{G}$ be an abelian group. The boundary map

$$C_\ell(K, \mathscr{G}) \xleftarrow{\partial} C_{\ell+1}(K, \mathscr{G})$$

is the group homomorphism defined by

$$\partial\left(\sum_s g_s \langle s \rangle\right) = \sum_s g_s\, \partial\langle s \rangle \quad (g_s \in \mathscr{G})$$

Lemma The boundary maps

$$C_{\ell-1}(K, \mathscr{G}) \xleftarrow{\partial} C_\ell(K, \mathscr{G}) \xleftarrow{\partial} C_{\ell+1}(K, \mathscr{G})$$

satisfy $\partial^2 = \partial \circ \partial = 0$.

Proof Since $\partial \circ \partial$ is linear, it suffices to check this on generators

$$\langle v_0, v_1, \ldots, v_{\ell+1} \rangle$$

as follows:

$$\begin{aligned}
\partial\left(\partial\langle v_0, v_1, \ldots, v_{\ell+1}\rangle\right) &= \partial\left[\sum_{j=0}^{\ell+1} (-1)^j \langle v_0, \ldots, \widehat{v_j}, \ldots, v_{\ell+1}\rangle\right] \\
&= \sum_{j=0}^{\ell+1} (-1)^j \partial\langle v_0, \ldots, \widehat{v_j}, \ldots, v_{\ell+1}\rangle \\
&= \sum_{j=0}^{\ell+1} (-1)^j \left[\sum_{i=0}^{j-1} (-1)^i \langle v_0, \ldots, \widehat{v_i}, \ldots, \widehat{v_j}, \ldots, v_{\ell+1}\rangle\right. \\
&\quad \left. + \sum_{i=j+1}^{\ell+1} (-1)^{i-1} \langle v_0, \ldots, \widehat{v_j}, \ldots, \widehat{v_i}, \ldots, v_{\ell+1}\rangle\right] \\
&= \sum_{i<j} (-1)^{i+j} \langle v_0, \ldots, \widehat{v_i}, \ldots, \widehat{v_j}, \ldots, v_{\ell+1}\rangle \\
&\quad + \sum_{i>j} (-1)^{i+j-1} \langle v_0, \ldots, \widehat{v_j}, \ldots, \widehat{v_i}, \ldots, v_{\ell+1}\rangle \\
&= \sum_{i<j} \left[(-1)^{i+j} + (-1)^{i+j-1}\right] \langle v_0, \ldots, \widehat{v_i}, \ldots, \widehat{v_j}, \ldots, v_{\ell+1}\rangle \\
&= 0.
\end{aligned}$$

Definition Given K and $\mathscr{G}$, let

$$\begin{aligned}
Z_\ell(K, \mathscr{G}) &= \{c \in C_\ell(K, \mathscr{G}) \mid \partial c = 0\}, \\
B_\ell(K, \mathscr{G}) &= \{\partial c \mid c \in C_{\ell+1}(K, \mathscr{G})\}, \\
H_\ell(K, \mathscr{G}) &= Z_\ell(K, \mathscr{G}) / B_\ell(K, \mathscr{G}).
\end{aligned}$$

Elements of $Z_\ell(K, \mathscr{G})$ are called *cycles,* and of $B_\ell(K, \mathscr{G})$ are called *boundaries.* The group $H_\ell(K, \mathscr{G})$ is called the ℓth *homology group* of K with coefficients in $\mathscr{G}$.

Remark It turns out that the groups $H_\ell(K, \mathscr{G})$ depend only on the topology of $[K]$. If $f : [K] \to [L]$ is a homeomorphism, then there is induced an isomorphism

$$f_* : H_\ell(K, \mathscr{G}) \to H_\ell(L, \mathscr{G}).$$

In particular, if K_1 and K_2 are simplicial complexes with $[K_1] = [K_2]$, then they have the same homology groups.

Exercise 15. Show that the vector space $H_0(K, \mathbb{R})$ has dimension equal to the number of connected components in $[K]$.

Example 1 Let K be the 1-skeleton of a 2-simplex; so K consists of three vertices v_0, v_1, v_2 three 1-simplices (v_0, v_1), (v_1, v_2) and (v_2, v_0) (see Figure 6.3). Then both $C_0(K, \mathbb{Z})$ and $C_1(K, \mathbb{Z})$ are isomorphic to $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$. $C_\ell(K, \mathbb{Z}) = 0$ for $\ell > 1$. A typical element c_1 of $C_1(K, \mathbb{Z})$ is of the form

$$c_1 = m_0\langle v_0, v_1\rangle + m_1\langle v_1, v_2\rangle + m_2\langle v_2, v_0\rangle \quad (m_0, m_1, m_2 \in \mathbb{Z}).$$

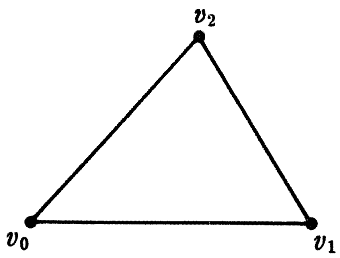


Figure 6.3

Its boundary ∂c_1 is given by

$$\begin{aligned} \partial c_1 &= m_0(\langle v_1\rangle - \langle v_0\rangle) + m_1(\langle v_2\rangle - \langle v_1\rangle) + m_2(\langle v_0\rangle - \langle v_2\rangle) \\ &= (m_2 - m_0)\langle v_0\rangle + (m_0 - m_1)\langle v_1\rangle + (m_1 - m_2)\langle v_2\rangle \end{aligned}$$

Thus $c_1 \in Z_1(K, \mathbb{Z})$ if and only if

$$m_2 - m_0 = 0, \quad m_0 - m_1 = 0, \quad m_1 - m_2 = 0, \iff m_0 = m_1 = m_2,$$

so

$$Z_1(K, \mathbb{Z}) = \{n(\langle v_0, v_1\rangle + \langle v_1, v_2\rangle + \langle v_2, v_0\rangle) \mid n \in \mathbb{Z}\} \cong \mathbb{Z}$$

Furthermore, $B_1(K, \mathbb{Z}) = 0$ because $C_2(K, \mathbb{Z}) = 0$. Hence

$$\boxed{H_1(K, \mathbb{Z}) = Z_1(K, \mathbb{Z})/B_1(K, \mathbb{Z}) \cong \mathbb{Z}.}$$

To compute $H_0(K, \mathbb{Z})$, note that a typical cycle $c_0 \in Z_0(K, \mathbb{Z}) = C_0(K, \mathbb{Z})$ is of the form

$$c_0 = n_0\langle v_0\rangle + n_1\langle v_1\rangle + n_2\langle v_2\rangle \quad (n_0, n_1, n_2 \in \mathbb{Z}).$$

Then $c_0 = \partial c_1$ for some

$$c_1 = m_0\langle v_0, v_1\rangle + m_1\langle v_1, v_2\rangle + m_2\langle v_2, v_0\rangle \in C_1(K, \mathbb{Z})$$

if and only if there exist (integer) solutions to the equations

$$m_2 - m_0 = n_0, \quad m_0 - m_1 = n_1, \quad m_1 - m_2 = n_2 \iff n_0 + n_1 + n_2 = 0.$$

Thus

$$B_0(K, \mathbb{Z}) = \{n_0\langle v_0\rangle + n_1\langle v_1\rangle + n_2\langle v_2\rangle \mid n_0 + n_1 + n_2 = 0\}.$$

Let $\varphi : Z_0(K, \mathbb{Z}) \to \mathbb{Z}$ be the homomorphism defined by

$$\varphi\left(n_0\langle v_0\rangle + n_1\langle v_1\rangle + n_2\langle v_2\rangle\right) = n_0 + n_1 + n_2.$$

Then the kernel of φ is just $B_0(K, \mathbb{Z})$; thus

$$\boxed{H_0(K, \mathbb{Z}) = Z_0(K, \mathbb{Z})/B_0(K, \mathbb{Z}) \cong \mathbb{Z}.}$$

Example 2 Let K be the complex consisting of all the faces of a 2-simplex (v_0, v_1, v_2). Then, as in Example 1,

$$\boxed{H_0(K, \mathbb{Z}) \cong \mathbb{Z}.}$$

Moreover, as before,

$$Z_1(K, \mathbb{Z}) = \{n\left(\langle v_0, v_1\rangle + \langle v_1, v_2\rangle + \langle v_2, v_0\rangle\right) \mid n \in \mathbb{Z}\}.$$

Now, however,

$$C_2(K, \mathbb{Z}) = \{n\langle v_0, v_1, v_2\rangle \mid n \in \mathbb{Z}\},$$

so that

$$\begin{aligned} B_1(K, \mathbb{Z}) &= \{\partial\left(n\langle v_0, v_1, v_2\rangle\right) \mid n \in \mathbb{Z}\} \\ &= \{n\left(\langle v_1, v_2\rangle - \langle v_0, v_2\rangle\right) + \langle v_0, v_1\rangle \mid n \in \mathbb{Z}\} \\ &= \{n\left(\langle v_0, v_1\rangle + \langle v_1, v_2\rangle + \langle v_2, v_0\rangle\right) \mid n \in \mathbb{Z}\} \\ &= Z_1(K, \mathbb{Z}) \end{aligned}$$

Hence

$$\boxed{H_1(K, \mathbb{Z}) = Z_1(K, \mathbb{Z})/B_1(K, \mathbb{Z}) = 0.}$$

Finally, since $\partial\left(n\langle v_0, v_1, v_2\rangle\right) = 0$ if and only if $n = 0$, $Z_2(K, \mathbb{Z}) = 0$, and hence

$$\boxed{H_2(K, \mathbb{Z}) = 0.}$$

Definition Let K be a simplicial complex. The ℓth Betti number β_ℓ of K is the integer

$$\beta_\ell = \dim H_\ell(K, \mathbb{R}).$$

The *Euler characteristic* $\chi(K)$ of K is the integer

$$\chi(K) = \sum_{\ell=0}^{\dim K} (-1)^\ell \beta_\ell.$$

Theorem Let K be a simplicial complex. For each ℓ with $0 \le \ell \le \dim K$, let α_ℓ denote the number of ℓ-simplices in K. Then

$$\chi(K) = \sum_{\ell=0}^{\dim K} (-1)^\ell \alpha_\ell;$$

that is, $\chi(K)$ is equal to the number of vertices $-$ the number of edges $+$ the number of 2-faces $\ldots$.

Proof For each ℓ, $0 \le \ell \le \dim K$, consider the linear map

$$C_{\ell-1}(K, \mathbb{R}) \xleftarrow{\partial} C_\ell(K, \mathbb{R}),$$

where C_{-1} is by definition the zero space. Then, by the rank and nullity theorem of linear algebra,

$$\begin{aligned} \alpha_\ell &= \dim C_\ell(K, \mathbb{R}) = \dim \operatorname{Ker} \partial + \dim \operatorname{Im} \partial \\ &= \dim Z_\ell(K, \mathbb{R}) + \dim B_{\ell-1}(K, \mathbb{R}) \quad (\ell = 0, 1, \ldots, \dim K). \end{aligned}$$

Moreover,

$$\begin{aligned} \beta_\ell &= \dim H_\ell(K, \mathbb{R}) = \dim \left[Z_\ell(K, \mathbb{R}) / B_\ell(K, \mathbb{R}) \right] \\ &= \dim Z_\ell(K, \mathbb{R}) - \dim B_\ell(K, \mathbb{R}). \end{aligned}$$

Thus

$$\begin{aligned} \chi(K) &= \sum_{\ell=0}^{\dim K} (-1)^\ell \beta_\ell = \sum_{\ell=0}^{\dim K} (-1)^\ell \left[\dim Z_\ell(K, \mathbb{R}) - \dim B_\ell(K, \mathbb{R}) \right] \\ &= \sum_{\ell=0}^{\dim K} (-1)^\ell \dim Z_\ell(K, \mathbb{R}) + \sum_{\ell=0}^{\dim K} (-1)^{\ell+1} \dim B_\ell(K, \mathbb{R}) \\ &= \sum_{\ell=0}^{\dim K} (-1)^\ell \dim Z_\ell(K, \mathbb{R}) + \sum_{\ell=1}^{\dim K} (-1)^\ell \dim B_{\ell-1}(K, \mathbb{R}) \quad (\text{since } \dim B_\ell = 0 \text{ for } \ell = \dim K) \\ &= \sum_{\ell=0}^{\dim K} (-1)^\ell \left[\dim Z_\ell(K, \mathbb{R}) + \dim B_{\ell-1}(K, \mathbb{R}) \right] \\ &= \sum_{\ell=0}^{\dim K} (-1)^\ell \alpha_\ell \end{aligned}$$

Remark If $[K]$ is homeomorphic to a connected compact orientable 2-dimensional manifold, then it turns out that $\beta_0 = 1$ and $\beta_2 = 1$, so that

$$\chi(K) = \beta_0 - \beta_1 + \beta_2 = 2 - \beta_1 \iff \beta_1 = 2 - \chi(K).$$

Furthermore, β_1 is always even for such K. It can be shown that any such surface is homeomorphic to a sphere with a certain number of "handles" attached; $\dfrac{1}{2}\beta_1$ is just the number of handles (see Figure 6.4)

Thus the homology groups completely determine the homeomorphism class of connected compact orientable surfaces. However, for higher dimensional manifolds, the homology groups contain comparatively little information.

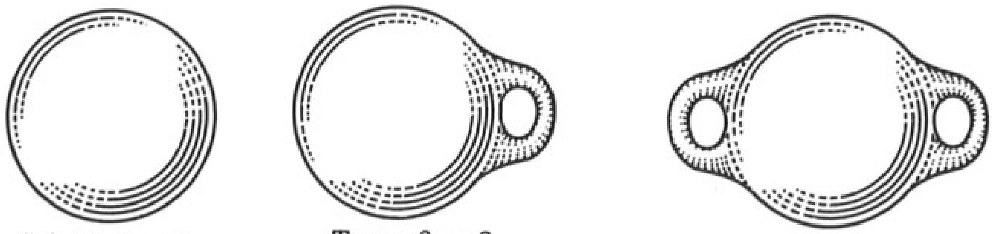


Figure 6.4

Remark We have been discussing a *homology* theory for simplicial complexes, that is, a theory arising from a sequence of groups and homomorphisms

$$\cdots \xleftarrow{\partial} C_{\ell-1}(K,\, \mathbb{R}) \xleftarrow{\partial} C_\ell(K,\, \mathbb{R}) \xleftarrow{\partial} C_{\ell+1}(K,\, \mathbb{R}) \longleftarrow \cdots .$$

where the map ∂ lowers the dimension of chains. On the other hand, in studying *De Rham cohomology*, we used a sequence

$$\cdots \xrightarrow{d} C^\infty(X,\, \Lambda^{\ell-1}(X)) \xrightarrow{d} C^\infty(X,\, \Lambda^{\ell}(X)) \xrightarrow{d} C^\infty(X,\, \Lambda^{\ell+1}(X)) \longrightarrow \cdots .$$

where the map d raised dimension (degree). In order to compare these two theories, it is convenient to define a *simplicial cohomology theory*. This is done by passing to dual spaces.

Definition Let K be a simplicial complex. For $0 \le \ell \le \dim K$, let

$$C^\ell(K) = [C_\ell(K,\, \mathbb{R})]^*.$$

Let $\partial^* : C^\ell(K) \to C^{\ell+1}(K)$ be the adjoint of the map $\partial : C_{\ell+1}(K,\, \mathbb{R}) \to C_\ell(K,\, \mathbb{R})$. Thus ∂^* is defined by

$$[\partial^*(\varphi)](c) = \varphi(\partial c) \quad \text{for each } \varphi \in C^\ell(K), \text{ and for each } c \in C_{\ell+1}(K,\, \mathbb{R}).$$

Then we get a sequence

$$\cdots \longrightarrow C^{\ell-1}(K) \xrightarrow{\partial^*} C^\ell(K) \xrightarrow{\partial^*} C^{\ell+1}(K) \longrightarrow \cdots .$$

Moreover, $\partial^* \circ \partial^* = 0$ since $\partial \circ \partial = 0$. Let

$$\begin{aligned} Z^\ell(K) &= \left\{\varphi \in C^\ell(K) \mid \partial^*\varphi = 0\right\}, \\ B^\ell(K) &= \left\{\partial^*\varphi \mid \varphi \in C^{\ell-1}(K)\right\}, \\ H^\ell(K) &= Z^\ell(K)/B^\ell(K). \end{aligned}$$

Elements of $C^\ell(K)$ are called *cochains*; elements of $Z^\ell(K)$ are *cocycles*; elements of $B^\ell(K)$ are *coboundaries*. The map ∂^* is the *coboundary operator*. $H^\ell(K)$ is the ℓth *cohomology group* of K.

Exercise 16. Verify that $H^\ell(K)$ is isomorphic to $[H_\ell(K,\, \mathbb{R})]^*$.

We shall need an explicit formula exhibiting the effect of the coboundary operator ∂^*. For each oriented ℓ-simplex $\langle s\rangle$ of K, let $\varphi_{\langle s\rangle} \in C^\ell(K)$ be defined by

$$\varphi_{\langle s\rangle}\langle t\rangle = \begin{cases} 1 & \text{if } \langle t\rangle = \langle s\rangle \\ -1 & \text{if } \langle t\rangle = -\langle s\rangle \\ 0 & \text{if } \langle t\rangle \ne \pm\langle s\rangle \end{cases}$$

Thus, if $\{\langle s_1\rangle, \ldots, \langle s_m\rangle\}$ is a basis for $C_\ell(K, \mathbb{R})$ (so that $\{s_1, \ldots, s_m\}$ is the set of all ℓ-simplices of K), then $\{\varphi_{\langle s_1\rangle}, \ldots, \varphi_{\langle s_m\rangle}\}$ is the dual basis for $C^\ell(K)$. Since ∂^* is linear, we need only compute the effect of ∂^* on these generators $\varphi_{\langle s\rangle}$.

Lemma

$$\boxed{\partial^* \varphi_{\langle v_0, \ldots, v_\ell\rangle} = \sum_v{}' \varphi_{\langle v, v_0, \ldots, v_\ell\rangle},}$$

where $\sum_v{}'$ denotes the sum over all vertices $v \in K$ such that $(v, v_0, \ldots, v_\ell)$ is an $(\ell+1)$-simplex of K.

Proof We need only check this formula on oriented $(\ell+1)$-simplices

$$\langle t\rangle = \langle w_0, w_1, \ldots, w_{\ell+1}\rangle$$

of K. If we set $\langle s\rangle = \langle v_0, v_1, \ldots, v_\ell\rangle$, the left side yields

$$\begin{aligned}
\left(\partial^* \varphi_{\langle s\rangle}\right)(\langle t\rangle) &= \varphi_{\langle s\rangle}\left(\partial\langle t\rangle\right) \\
&= \varphi_{\langle s\rangle}\left(\sum_{i=0}^{\ell+1}(-1)^i \langle w_0, \ldots, \widehat{w}_i, \ldots, w_{\ell+1}\rangle\right) \\
&= \sum_{i=0}^{\ell+1}(-1)^i \varphi_{\langle s\rangle}\left(\langle w_0, \ldots, \widehat{w}_i, \ldots, w_{\ell+1}\rangle\right)
\end{aligned}$$

But each term of this sum is zero unless, for some j, $(w_0, \ldots, \widehat{w_j}, \ldots, w_{\ell+1}) = (s)$; that is, unless (s) is a face of (t). If (s) is a face of (t), then $(t) = (v, v_0, \ldots, v_\ell)$ for some vertex $v \in K$, in which case either

(1) $\langle t\rangle = \langle v, v_0, \ldots, v_\ell\rangle$ and $\left(\partial^* \varphi_{\langle s\rangle}\right)(\langle t\rangle) = 1$; or
(2) $\langle t\rangle = -\langle v, v_0, \ldots, v_\ell\rangle$ and $\left(\partial^* \varphi_{\langle s\rangle}\right)(\langle t\rangle) = -1$.

Thus

$$\begin{aligned}
\left(\partial^* \varphi_{\langle s\rangle}\right)(\langle t\rangle) &= \begin{cases} 1 & \text{if } \langle t\rangle = \langle v, v_0, \ldots, v_\ell\rangle \text{ for some } v \\ -1 & \text{if } \langle t\rangle = -\langle v, v_0, \ldots, v_\ell\rangle \text{ for some } v \\ 0 & \text{in all other cases} \end{cases} \\
&= \left(\sum_v{}' \varphi_{\langle v, v_0, \ldots, v_\ell\rangle}\right)(\langle t\rangle)
\end{aligned}$$

Since this holds for arbitrary $\langle t\rangle$, the formula is established.

§6.2 De Rham's Theorem

Definition A *smoothly triangulated manifold* is a triple (X, K, h), where X is a C^∞ manifold, K is a simplicial complex, and $h : [K] \to X$ is a homeomorphism such that for each simplex s of K, the map $h|_{[s]} : [s] \to X$ has an extension $h|_s$ to a neighborhood U of $[s]$ in the plane of $[s]$ such that $h|_s : U \to X$ is a smooth submanifold.

Remark If dim $X = n$, we need only require that this last condition be satisfied for each n-simplex of K, since every simplex of K is a face of an n-simplex and since restrictions of smooth maps to submanifolds are smooth.

Example Let $X = S^n$. Let K be the n-skeleton of an $(n+1)$-simplex circumscribed about S^n. Let $h : [K] \to S^n$ be radial projection. Then (X, K, h) is a smoothly triangulated manifold (Figure 6.5).

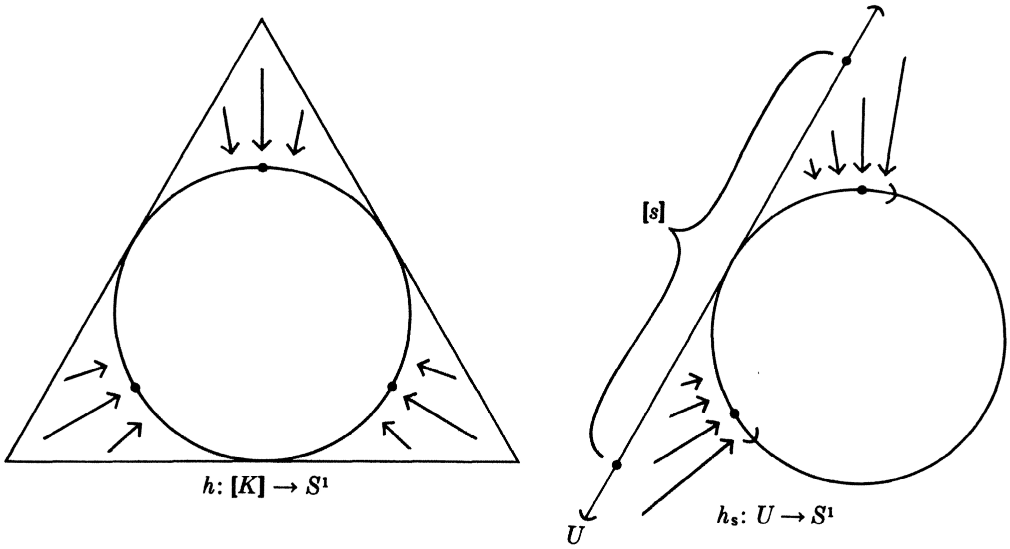


Figure 6.5

Remark It can be shown that every compact smooth manifold can be smoothly triangulated. The proof is difficult and will not be presented here. Note that smoothly triangulated manifolds are compact because $[K]$ is compact for each (finite) simplicial complex K.

The goal of this section is to show that for smoothly triangulated manifolds (X, K, h), the De Rham cohomology of X is isomorphic to the simplicial cohomology of K. For this, we shall need the following facts about barycentric coordinates. Recall that we have previously discussed the barycentric coordinates of a point relative to the vertices of a simplex containing it. We now extend this concept.

Definition Let K be a simplicial complex and let v be a vertex of K. The *star of v*, denoted $\mathrm{St}\,(v)$, is the point set

$$\mathrm{St}\,(v) = \bigcup_{\substack{v \in [s] \\ (s) \in K}} (s) \quad \text{(an open set in } [K] \text{ containing } v, \text{ and } v \text{ is the only vertex of } K \text{ lies in } \mathrm{St}\,(v))$$

Definition Let K be a simplicial complex and let $v_1, \ldots, v_m$ denote the vertices of K. Suppose $x \in [K]$. For $1 \le j \le m$, the jth *barycentric coordinate* $b_j(x)$ of x is defined as follows. If $x \notin \mathrm{St}\,(v_j)$, then $b_j(x) = 0$; if $x \in \mathrm{St}\,(v_j)$, then $x \in (s)$ for some simplex s having v_j as a vertex, and $b_j(x)$ is equal to the barycentric coordinate of x in s relative to the vertex v_j.

Remark The following facts are easily verified.

(1) $b_j : [K] \to \mathbb{R}$ is a continuous function.

(2) $b_j(x) \ge 0$ for all $1 \le j \le m$ and $\displaystyle\sum_{j=1}^{m} b_j(x) = 1$ for each $x \in [K]$.

(3) $x = \sum_{j=1}^{m} b_j(x) v_j$.

(4) $b_{j_0}(x) \neq 0$, $b_{j_1}(x) \neq 0, \ldots, b_{j_\ell}(x) \neq 0$ for some $x \in [K]$ if and only if $v_{j_0}, \ldots, v_{j_\ell}$ are the vertices of an ℓ-simplex of K.

Definition Let K be a simplicial complex, and let s be a simplex of K. The *star of s, denoted $St(s)$,* is the union of all the open simplices (t) of K such that (s) is a face of (t).

Remarks The following facts are easily verified.

(1) For $s = v$ a 0-simplex (i.e. a vertex) of K, $\mathrm{St}\,(s) = \mathrm{St}\,(v)$, as defined above.

(2) $\mathrm{St}\,(s)$ is an open set in $[K]$. (This is an elementary consequence of (3).)

(3) If $(s) = (v_{j_0}, \ldots, v_{j_\ell})$ and $x \in [K]$, then $x \in \mathrm{St}\,(s)$ if and only if $b_{j_i}(x) \neq 0$ for all $0 \leq i \leq \ell$.

(4) If $(s) = (v_{j_0}, \ldots, v_{j_\ell})$, then

$$[K] \setminus \mathrm{St}\,(s) = \{x \in [K]; b_{j_i}(x) = 0 \text{ for some } 0 \leq i \leq \ell\}.$$

(5) If s_1 and s are ℓ-simplices of K with $s_1 \neq s$, then $[s_1] \subset [K] \setminus \mathrm{St}\,(s)$.

Given a smoothly triangulated manifold (X, K, h), we want to define, for each ℓ, an isomorphism from $H^\ell(X, d)$ onto $H^\ell(K)$. To do this, note that homomorphisms $\tilde{f}_\ell : H^\ell(X, d) \to H^\ell(K)$ are defined whenever there is given a sequence of linear maps $f_\ell : C^\infty(X, \Lambda^\ell(X)) \to C^\ell(K)$ such that $\partial^* \circ f_\ell = f_{\ell+1} \circ d$ for all ℓ.

$$\begin{array}{ccccccc}
\cdots \to & C^\infty(X, \Lambda^\ell(X)) & \xrightarrow{d} & C^\infty(X, \Lambda^{\ell+1}(X)) & \to \cdots \\
& \big\downarrow f_\ell & & \big\downarrow f_{\ell+1} & \\
\cdots \to & C^\ell(K) & \xrightarrow{\partial^*} & C^{\ell+1}(K) & \to \cdots
\end{array}$$

For then $f_\ell(Z^\ell(X, d)) \subset Z^\ell(K)$, because $d\omega = 0$ $(\omega \in C^\ell(X, d))$ implies that

$$\partial^* (f_\ell(\omega)) = f_{\ell+1}(d\omega) = f_{\ell+1}(0) = 0.$$

Also $f_\ell(B^\ell(X, d)) \subset B^\ell(K)$, because $\omega = d\tau$ $(\tau \in C^{\ell-1}(X, d))$ implies that

$$f_\ell(\omega) = f_\ell(d\tau) = \partial^* (f_{\ell-1}(\tau)) \in \operatorname{Im} \partial^*.$$

Thus f_ℓ induces

$$\tilde{f}_\ell : H^\ell(X, d) = Z^\ell(X, d)/B^\ell(X, d) \to Z^\ell(K)/B^\ell(K) = H^\ell(K).$$

We now proceed to define such a sequence of linear maps

$$\int_\ell : C^\infty(X, \Lambda^\ell(X)) \to C^\ell(K).$$

For $\omega \in C^\infty(X, \Lambda^\ell(X))$, $\displaystyle\int_\ell(\omega)$ will be a linear functional on $C_\ell(K)$. Thus it suffices to specify the values of $\displaystyle\int_\ell(\omega)$ on basis elements of $C_\ell(K)$, that is, on oriented ℓ-simplices $\langle s \rangle$. To do this, consider the smooth map $h_s : U \to X$. Then $h_s^*(\omega)$ is a smooth ℓ-form on U, an open set in the plane of $[s]$; that is, in an ℓ-dimensional Euclidean space. We define $\int_\ell (\omega)\,(\langle s\rangle)$ to be the integral of this ℓ-form over $\langle s\rangle$:

$$\int_\ell (\omega)\,(\langle s\rangle) = \int_{\langle s\rangle} h_s^*(\omega).$$

In other words, let $(r_1, \ldots, r_\ell)$ denote coordinates in the plane of $[s]$ consistent with the orientation of $\langle s\rangle$; so if $\langle s\rangle = \langle v_0, \ldots, v_\ell\rangle$, let $(r_1, \ldots, r_\ell)$ be coordinates relative to the ordered basis $\{v_1 - v_0, \ldots, v_\ell - v_0\}$. Then

$$h_s^*(\omega) = g\,dr_1 \wedge \cdots \wedge dr_\ell \quad \text{for some continuous function } g \text{ on } U,$$

and

$$\int_\ell (\omega)\,(\langle s\rangle) = \int_{[s]} g\,dr_1 \cdots dr_\ell \quad \text{(Riemann integral)}.$$

Note that this integral is independent of the homeomorphism h; that is, it depends only on the point set $h([s])$ and its orientation by the change of variables theorem for integrals.

$$\boxed{\text{Claim} \ : \quad \partial^* \circ \int_\ell = \int_{\ell+1} \circ\, d.}$$

This is just Stokes's theorem. For given any smooth ℓ-form ω and oriented $(\ell+1)$-simplex $\langle s\rangle$,

$$\begin{aligned}\left[\int_{\ell+1} \circ\, d(\omega)\right](\langle s\rangle) &= \int_{\langle s\rangle} (h_s)^*\,(d\omega) = \int_{\langle s\rangle} d\left(h_s^*\,(\omega)\right) = \int_{\partial\langle s\rangle} h_s^*\,(\omega) \quad \text{(by Stokes's theorem)}\\ &= \int_\ell (\omega)\,(\partial\langle s\rangle) = \left[\partial^* \circ \int_\ell (\omega)\right]\langle s\rangle.\end{aligned}$$

Thus $\int_\ell$ induces a homomorphism $\tilde{\int_\ell} : H^\ell(X, d) \to H^\ell(K)$.

Theorem (De Rham's Theorem) Let (X, K, h) be a smoothly triangulated manifold. Then

$$\tilde{\int_\ell} : H^\ell(X, d) \to H^\ell(K)$$

is an isomorphism for each ℓ $(0 \le \ell \le \dim X)$.

Remark To define the inverse of $\tilde{\int_\ell}$ for each ℓ, we construct a special partition of unity, subordinate to the open covering

$$\{\mathrm{St}\,(v) \mid v \text{ is a vertex of } K\}$$

of X. Let $v_1, \ldots, v_m$ denote the vertices of K. For each $1 \le j \le m$, let b_j be the jth barycentric coordinate function on $[K] = X$ and let $\dim X = n$,

$$F_j = \left\{x \in X \mid b_j(x) \ge \frac{1}{n+1}\right\}, \quad G_j = \left\{x \in X \mid b_j(x) \le \frac{1}{n+2}\right\}.$$

Then F_j and G_j are closed sets in X with the following properties (see Figure 6.6).

(1) $F_j \subset \mathrm{St}\,(v_j)$.

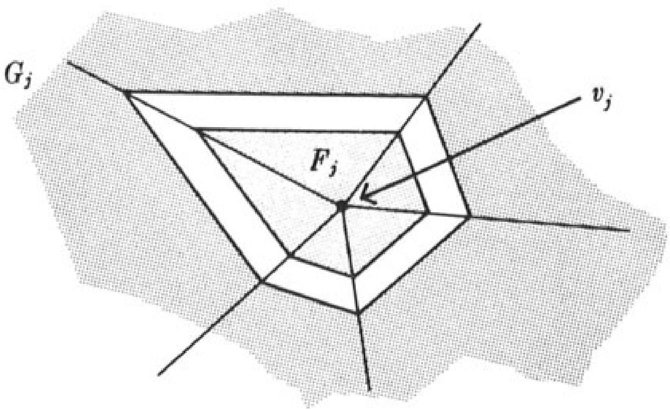


Figure 6.6

(2) $X \setminus \mathrm{St}\,(v_j) \subset G_j$.

(3) $F_j \cap G_j = \emptyset$; that is, $F_j \subset G'_j = X \setminus G_j$.

(4) There exists a smooth function $f_j \geq 0$ such that $f_j > 0$ on F_j, and $f_j = 0$ on G_j.
Proof F_j is a closed set in the compact space X, hence is compact; thus an $f_j \geq 0$ can be found which is greater than 0 on F_j and equal to 0 outside the open set $X \setminus G_j = G'_j \supset F_j$.

(5) The closed sets F_j cover X. In particular, for each $x \in X$, $f_j(x) \neq 0$ for some j. Furthermore, $X \setminus G_j = G'_j$ is an open covering of X.
Proof Given $x \in X$, $x \in (s)$ for some simplex $(s) = (v_{j_0}, \ldots, v_{j_\ell})$ of dimension $\ell \leq n$. Now $b_j(x) = 0$ for $j \notin \{j_0, \ldots, j_\ell\}$ and $\sum_{i=0}^{\ell} b_{j_i}(x) = 1$. Since $\ell + 1 \leq n + 1$, $b_{j_i}(x) \geq 1/(n+1)$ for some $0 \leq i \leq \ell$. Thus $x \in F_{j_i}$ for this $1 \leq j_i \leq m$.

(6) From (5), $\sum_{j=1}^{m} f_j > 0$, so that $g_j = f_j / \sum_{k=1}^{m} f_k$ is defined and smooth on X. Furthermore, $\{g_j\}$ is a smooth partition of unity on X subordinate to $\{X \setminus G_j = G'_j\}$; that is, $\sum_{j=1}^{m} g_j = 1$, and g_j vanishes outside G'_j. Since $G'_j \subset \mathrm{St}\,(v_j)$, the partition of unity $\{g_j\}$ is also subordinate to the open covering $\{\mathrm{St}\,(v_j)\}$.

Let $\left\{\langle s_i \rangle \mid 1 \leq i \leq \binom{m}{\ell+1}\right\}$ be a basis for $C_\ell(K, \mathbb{R})$, and let $\left\{\varphi_{\langle s_i \rangle} \mid 1 \leq i \leq \binom{m}{\ell+1}\right\}$ be the dual basis for $C^\ell(K)$. To define the inverse linear map $\alpha_\ell : C^\ell(K) \to C^\infty(X, \Lambda^\ell(X))$, it suffices to specify the values of α_ℓ on the generators $\varphi_{\langle s \rangle}$ of $C^\ell(K)$.

Definition For each $0 \leq \ell \leq \dim X$, for each oriented ℓ-simplex (basis element) $\langle s \rangle = \langle v_{j_0}, \ldots, v_{j_\ell} \rangle$ of $C_\ell(K)$, $\alpha_\ell\left(\varphi_{\langle s \rangle}\right)$ is the ℓ-form defined by

$$\alpha_\ell\left(\varphi_{\langle s \rangle}\right) = \ell! \sum_{i=0}^{\ell} (-1)^i g_{j_i}\, dg_{j_0} \wedge \cdots \wedge \widehat{dg_{j_i}} \wedge \cdots \wedge dg_{j_\ell}\,,$$

where $\varphi_{\langle s \rangle}$ is the dual basis element of $\langle s \rangle$ in $C^\ell(K)$, and $\{g_j\}$ are the smooth functions defined above.

De Rham's Theorem is a consequence of the following two lemmas.

Lemma 1 There exists a sequence of linear maps

$$\alpha_\ell : C^\ell(K) \to C^\infty(X,\, \Lambda^\ell(X)), \quad \text{for each } 0 \le \ell \le \dim X,$$

with the following properties.

(1) $d \circ \alpha_\ell = \alpha_{\ell+1} \circ \partial^*$.

(2) $\int_\ell \circ\, \alpha_\ell = \text{identity}$.

(3) If c^0 denotes the 0-cochain such that $c^0(v) = 1$ for each vertex v in K, then $\alpha_0(c^0) = 1$; that is, $\alpha_0(c^0)$ is the 0-form equal to the constant function 1.

(4) If $\langle s \rangle$ is an oriented ℓ-simplex of K, then the $\alpha_\ell\left(\varphi_{\langle s \rangle}\right)$ is identically zero in a neighborhood of $X \setminus \mathrm{St}\,(s)$.

Lemma 2 Let ω be a closed ℓ-form on X. Suppose $\int_\ell(\omega) = \partial^* c$ for some $c \in C^{\ell-1}(K)$. Then there exists an $(\ell-1)$-form τ on X such that $\int_{\ell-1}(\tau) = c$ and $d\tau = \omega$.

Remark Lemma 1 shows that $\tilde{\int}_\ell$ is surjective. For given $z \in Z^\ell(K)$, let $\omega = \alpha_\ell(z)$. Then $\omega \in Z^\ell(X,\, d)$ because

$$d\omega = d \circ \alpha_\ell(z) = \alpha_{\ell+1}(z) \circ \partial^*(z) = \alpha_{\ell+1}(0) = 0.$$

Furthermore, $\tilde{\int}_\ell(\omega) = \tilde{\int}_\ell \circ\, \alpha_\ell(z) = z$. Thus $\int_\ell : Z^\ell(X,\, d) \to Z^\ell(K)$ is surjective; hence so is $\tilde{\int}_\ell$. (Note that Property (1) says that the map α_ℓ induces a homomorphism $\tilde{\alpha}_\ell : H^\ell(K) \to H^\ell(X,\, d)$. Property (2) says that this map is a right inverse to $\tilde{\int}_\ell$.)

Lemma 2 shows that $\tilde{\int}_\ell$ is injective. For if $\omega \in Z^\ell(X,\, d)$ and $\int_\ell(\omega) \in B^\ell(K)$, then $\omega \in B^\ell(X,\, d)$ by Lemma 2.

Thus Lemmas 1 and 2 together do, as claimed, imply De Rham's theorem.

Proof of Lemma 1 For notational convenience, we shall identity $[K]$ and X through the homeomorphism h; that is, we shall assume that $[K] = X$ and that $h = \text{identity}$.

Verification of Property (1). Clearly,

$$d \circ \alpha_\ell\left(\varphi_{\langle s \rangle}\right) = (\ell+1)!\, dg_{j_0} \wedge \cdots \wedge dg_{j_\ell}$$

On the other hand,

$$\begin{aligned}
\alpha_{\ell+1} \circ \partial^*\left(\varphi_{\langle s \rangle}\right) &= \alpha_{\ell+1}\left(\sum_{v_k}{}' \varphi_{\langle v_k,\, v_{j_0}, \ldots,\, v_{j_\ell} \rangle}\right) \\
&= (\ell+1)! \sum_k{}' \left[g_k\, dg_{j_0} \wedge \cdots \wedge dg_{j_\ell} - \sum_{i=0}^{\ell} (-1)^i g_{j_i}\, dg_{j_0} \wedge \cdots \wedge \widehat{dg_{j_i}} \wedge \cdots \wedge dg_{j_\ell}\right].
\end{aligned}$$

Claim If the vertices $v_k,\, v_{j_0}, \ldots,\, v_{j_\ell}$ are distinct and yet are not the vertices of an $(\ell+1)$-simplex of K, then

$$g_k\, dg_{j_0} \wedge \cdots \wedge dg_{j_\ell} \equiv 0.$$

For, if $x \notin \mathrm{St}(v_k)$, then $g_k(x) = 0$. If $x \in \mathrm{St}(v_k)$, then $b_k(x) \neq 0$. But now $b_{j_i}(x) = 0$ for some $0 \leq i \leq \ell$, for otherwise $b_k(x) \neq 0$, $b_{j_0}(x) \neq 0, \ldots, b_{j_\ell}(x) \neq 0$, hence $(v_k, v_{j_0}, \ldots, v_{j_\ell})$ is an $(\ell+1)$-simplex. But this is a contradiction. Using this i, let

$$U = \left\{ y \in X \mid b_{j_i}(y) < \frac{1}{n+2} \right\}.$$

Then U is an open set in X containing x, and g_{j_i} is identically zero on U because $U \subset G_{j_i}$. Hence $dg_{j_i} \equiv 0$ on U, and, in particular, $dg_{j_i}(x) = 0$. This completes the proof of the claim.

Applying this result to the terms of the above expression for $\alpha_{\ell+1} \circ \partial^* \left(\varphi_{\langle s \rangle}\right)$ yields

$$(A) \qquad {\sum_k}' g_k \, dg_{j_0} \wedge \cdots \wedge dg_{j_\ell} = \sum_{k \notin \{j_0, \ldots, j_\ell\}} g_k \, dg_{j_0} \wedge \cdots \wedge dg_{j_\ell},$$

since those terms on the right-hand side which do not appear on the left are identically zero; and

$$\begin{aligned}
(B) \qquad & {\sum_k}' \sum_{i=0}^{\ell} (-1)^i g_{j_i} \, dg_k \wedge dg_{j_0} \wedge \cdots \wedge \widehat{dg_{j_i}} \wedge \cdots \wedge dg_{j_\ell} \\
= \; & \sum_{i=0}^{\ell} (-1)^i {\sum_k}' g_{j_i} \, dg_k \wedge dg_{j_0} \wedge \cdots \wedge \widehat{dg_{j_i}} \wedge \cdots \wedge dg_{j_\ell} \\
= \; & \sum_{i=0}^{\ell} (-1)^i \sum_{k \notin \{j_0, \ldots, j_\ell\}} g_{j_i} \, dg_k \wedge dg_{j_0} \wedge \cdots \wedge \widehat{dg_{j_i}} \wedge \cdots \wedge dg_{j_\ell} \\
= \; & \sum_{i=0}^{\ell} (-1)^i \sum_{k \neq j_i} g_{j_i} \, dg_k \wedge dg_{j_0} \wedge \cdots \wedge \widehat{dg_{j_i}} \wedge \cdots \wedge dg_{j_\ell} \\
= \; & \sum_{i=0}^{\ell} (-1)^i g_{j_i} \left(\sum_{k \neq j_i} dg_k \right) \wedge dg_{j_0} \wedge \cdots \wedge \widehat{dg_{j_i}} \wedge \cdots \wedge dg_{j_\ell} \\
= \; & \sum_{i=0}^{\ell} (-1)^i g_{j_i} \left(-dg_{j_i} \right) \wedge dg_{j_0} \wedge \cdots \wedge \widehat{dg_{j_i}} \wedge \cdots \wedge dg_{j_\ell} \\
= \; & -\sum_{i=0}^{\ell} g_{j_i} \, dg_{j_0} \wedge \cdots \wedge dg_{j_\ell} \quad \left(\text{since } \sum_{k=1}^{m} g_k = 1 \implies \sum_{k=1}^{m} dg_k = 0\right).
\end{aligned}$$

Hence, substituting (B) from (A),

$$\begin{aligned}
\alpha_{\ell+1} \circ \partial^* \left(\varphi_{\langle s \rangle}\right) &= (\ell+1)! \left(\sum_{k=1}^{m} g_k \right) dg_{j_0} \wedge \cdots \wedge dg_{j_\ell} \\
&= (\ell+1)! \, dg_{j_0} \wedge \cdots \wedge dg_{j_\ell} \\
&= d \circ \alpha_\ell \left(\varphi_{\langle s \rangle}\right).
\end{aligned}$$

Verification of Property (3). Since $\alpha_0 \left(\varphi_{\langle v_j \rangle}\right) = g_j$,

$$\alpha_0 \left(c^0\right) = \alpha_0 \left(\sum_{j=1}^{m} \varphi_{\langle v_j \rangle} \right) = \sum_{j=1}^{m} g_j = 1.$$

Verification of Property (4). Suppose $\langle s\rangle = \langle v_{j_0}, \ldots, v_{j_\ell}\rangle$. Then

$$\alpha_\ell\left(\varphi_{\langle s\rangle}\right) = \ell! \sum_{i=0}^{\ell} (-1)^i g_{j_i}\, dg_{j_0} \wedge \cdots \wedge \widehat{dg_{j_i}} \wedge \cdots \wedge dg_{j_\ell}.$$

Note that if $x \in X$ is such that $b_{j_k}(x) < \dfrac{1}{n+2}$ for some $0 \le k \le \ell$, then $x \in G_{j_k}$, so that g_{j_k} and dg_{j_k}, and hence $\alpha_\ell\left(\varphi_{\langle s\rangle}\right)$, are zero at x. Thus $\alpha_\ell\left(\varphi_{\langle s\rangle}\right)$ is identically zero on

$$\left\{x \in X \mid b_{j_i}(x) < \frac{1}{n+2} \text{ for some } 0 \le k \le \ell\right\},$$

which is an open set containing $X \setminus \mathrm{St}\,(s)$.

Verification of Property (2). For $\ell = 0$, $\displaystyle\int_0 \circ\, \alpha_0\left(\varphi_{\langle v_j\rangle}\right)$, $(j \in \{1, \ldots, m\})$ is the 0-cochain given by

$$\left[\int_0 \circ\, \alpha_0\left(\varphi_{\langle v_j\rangle}\right)\right](\langle v_k\rangle) = \left[\int_0 (g_j)\right]\langle v_k\rangle = g_j(v_k).$$

But note that $g_j(v_k) = 0$ for $k \neq j$ since $v_k \in \mathrm{St}\,(v_j)$ and $g_j = 0$ outside $\mathrm{St}\,(v_j)$. Furthermore,

$$1 = \sum_{j=1}^{m} g_j(v_k) = g_k(v_k) \quad \text{(for each } k\text{)}.$$

Hence

$$\begin{aligned}\left[\int_0 \circ\, \alpha_0\left(\varphi_{\langle v_j\rangle}\right)\right](\langle v_k\rangle) &= \begin{cases} 1 & (\text{if } k = j) \\ 0 & (\text{if } k \neq j)\end{cases} \\ &= \varphi_{\langle v_j\rangle}(\langle v_k\rangle)\end{aligned}$$

Since this holds for all j and k, $\displaystyle\int_0 \circ\, \alpha_0 = $ identity as required.

Now assume Property (2) for dimension $\ell - 1$. For $\langle s\rangle$ and $\langle s\rangle$ oriented ℓ-simplices of K,

$$\left[\int_\ell \circ\, \alpha_\ell\left(\varphi_{\langle s\rangle}\right)\right](\langle t\rangle) = \int_{\langle t\rangle} \alpha_\ell\left(\varphi_{\langle s\rangle}\right) \overset{\textbf{Claim}}{=} \begin{cases} 1 & \text{if } \langle s\rangle = \langle t\rangle \\ 0 & \text{if } \quad s \neq t\end{cases}.$$

Proof of Claim If $s \neq t$, then $\displaystyle\int_{\langle s\rangle} \alpha_\ell\left(\varphi_{\langle s\rangle}\right) = 0$ by **Property** (4) since $\alpha_\ell\left(\varphi_{\langle s\rangle}\right)$ is identically zero in a neighborhood of $X \setminus \mathrm{St}\,(s) \supset [t]$.

If $\langle s\rangle = \langle t\rangle$, let $\langle r\rangle = \langle v_{j_1}, \ldots, v_{j_\ell}\rangle$ and $\langle s\rangle = \langle v_{j_0}, v_{j_1}, \ldots, v_{j_\ell}\rangle$, then

$$\int_{\langle s\rangle} \alpha_\ell\left(\partial^* \varphi_{\langle r\rangle}\right) = \int_{\langle s\rangle} d\left[\alpha_{\ell-1}\left(\varphi_{\langle r\rangle}\right)\right] = \int_{\partial\langle s\rangle} \alpha_{\ell-1}\left(\varphi_{\langle r\rangle}\right).$$

But $\partial\langle s\rangle = \langle r\rangle$ plus an alternating sum of other oreiented $(\ell-1)$-simplices, so

$$\int_{\partial\langle s\rangle} \alpha_{\ell-1}\left(\varphi_{\langle r\rangle}\right) = \int_{\langle r\rangle} \alpha_{\ell-1}\left(\varphi_{\langle r\rangle}\right) = 1 \quad \text{by induction.}$$

Hence

$$1 = \int_{\langle s\rangle} \alpha_\ell \left(\partial^* \varphi_{\langle r\rangle}\right) = \int_{\langle s\rangle} \alpha_\ell \left(\varphi_{\langle s\rangle} + \text{terms of type } \varphi_{\langle t\rangle} \ (t \neq s)\right) = \int_{\langle s\rangle} \alpha_\ell \left(\varphi_{\langle s\rangle}\right).$$

In order to prove Lemma 2, we shall need the following lemma.

Lemma 3 Let s be a k-simplex in $\mathbb{R}^n$.

(a_r) Suppose $r \geq 0$ and $k \geq 1$. Let ω be a smooth closed r-form defined "near" $[s^{k-1}]$; that is, defined in a neighborhood of $[s^{k-1}]$. If $k = r+1$, assume further that $\int_{\partial\langle s\rangle} \omega = 0$. Then there exists a smooth closed r-form τ defined near $[s]$ such that $\tau = \omega$ near $[s^{k-1}]$.

(b_r) Suppose $r \geq 1$ and $k \geq 1$. Let ω be a smooth closed r-form defined near $[s]$. Suppose τ is a smooth $(r-1)$-form defined near $[s^{k-1}]$ such that $d\tau = \omega$ near $[s^{k-1}]$. If $k = r$, assume further that $\int_{\partial\langle s\rangle} \tau = \int_{\langle s\rangle} \omega$. Then there exists a smooth $(r-1)$-form τ' defined near $[s]$ such that $\tau' = \tau$ near $[s^{k-1}]$, and $d\tau' = \omega$ near $[s]$.

Remark That the integral conditions are necessary in (a_r) and (b_r) is a consequence of Stokes's theorem. For in (a_r), if τ exists, then

$$\int_{\partial\langle s\rangle} \omega = \int_{\partial\langle s\rangle} \tau = \int_{\langle s\rangle} d\tau = \int_{\langle s\rangle} 0 = 0;$$

and in (b_r), if τ' exists, then

$$\int_{\langle s\rangle} \omega = \int_{\langle s\rangle} d\tau' = \int_{\partial\langle s\rangle} \tau' = \int_{\partial\langle s\rangle} \tau.$$

Proof of Lemma 3 We shall first verify (a_0) and then establish that

$$(a_0) \implies (b_1) \implies (a_1) \implies (b_2) \implies \cdots.$$

Proof of (a_0) : ω is a smooth 0-form; that is, a smooth function defined near $[s^{k-1}]$; and $d\omega = 0$. Hence ω is constant on the components of its domain. If $k > 1$, $[s^{k-1}]$ is connected, so ω is a constant function c in a neighborhood of $[s^{k-1}]$. Set $\tau = c$ in a neighborhood of $[s]$. If $k = 1$, then $\langle s\rangle = \langle v_0, v_1\rangle$ for some pair of vertices v_0, v_1, and

$$0 = \int_{\partial\langle s\rangle} \omega = \omega(v_1) - \omega(v_0).$$

Thus the constant value of ω near v_1 equals the constant value of ω near v_0; that is, ω is constant near $[s^{k-1}]$ as before. Once again, set τ equal to this constant function on a neighborhood of $[s]$.

Proof of $(a_{r-1}) \implies (b_r)$: Let ω be a closed r-form ($r \geq 1$) defined on an open set containing $[s]$. Since any open set containing $[s]$ must contain another open set about $[s]$ which is diffeomorphic (smoothly homeomorphic with a smooth inverse) to an open ball, ω is exact near $[s]$ by Poincaré's Lemma (Section 5.2); that is, there exists a smooth $(r-1)$-form τ_1 defined near $[s]$ such that $d\tau_1 = \omega$ near $[s]$. Now in general, τ_1 will not be equal to τ near $[s^{k-1}]$. Consider the difference $\tau_1 - \tau$ near $[s^{k-1}]$. It is closed since, near $[s^{k-1}]$,

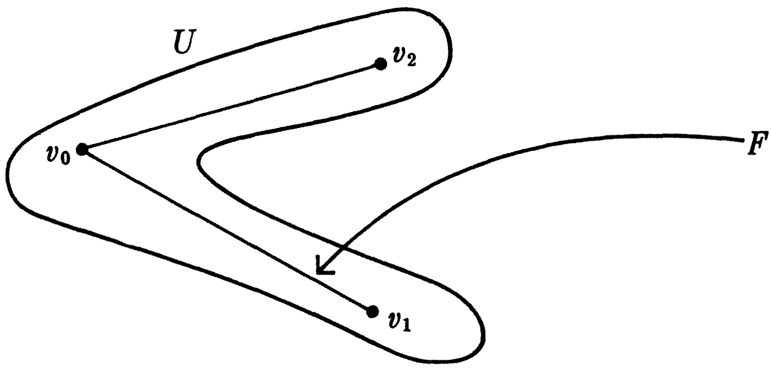
$$d\left(\tau_1 - \tau\right) = \omega - \omega = 0.$$

Furthermore, if $k = (r-1)+1 = r$, then

$$\int_{\partial\langle s\rangle} (\tau_1 - \tau) = \int_{\partial\langle s\rangle} \tau_1 - \int_{\partial\langle s\rangle} \tau = \int_{\langle s\rangle} d\tau_1 - \int_{\partial\langle s\rangle} \tau = \int_{\langle s\rangle} \omega - \int_{\partial\langle s\rangle} \tau = 0 \quad \text{(by hypothesis)}.$$

Thus we can apply (a_{r-1}) to the form $\tau_1 - \tau$. There exists a smooth closed $(r-1)$-form μ defined near $[s]$ such that $\mu = \tau_1 - \tau$ near $[s^{k-1}]$. Let $\tau' = \tau_1 - \mu$. Then τ' is a smooth $(r-1)$-form defined near $[s]$ such that $\tau' = \tau_1 - \mu = \tau$ near $[s^{k-1}]$, and $d\tau' = d\tau_1 - d\mu = \omega - 0 = \omega$ near $[s]$.

Proof of $(b_r) \implies (a_r)$: $\langle s\rangle = \langle v_0, \ldots, v_k\rangle$ for some choices of vertices $v_0, \ldots, v_k$; let $\langle t\rangle = \langle v_1, \ldots, v_k\rangle$. Let $F = [s^{k-1}] \setminus (t)$. Since F is contained in a star-shaped open neighborhood U of F (see Figure 6.7) and ω is closed near F, there exists a smooth $(r-1)$-form μ defined near F such that $d\mu = \omega$ near F by the Poincaré's Lemma. In particular, $d\mu = \omega$ near $[t^{k-2}]$.



$<s> = <v_0,v_1,v_2>$, $\quad <t> = <v_1,v_2>$

Figure 6.7

If $k > 1$, we would like to apply (b_r) to the forms ω and μ and the $(k-1)$-simplex t. In order to do this we must check if $k-1 = r$, then $\int_{\langle t\rangle} \omega - \int_{\partial\langle t\rangle} \mu = 0$. But, letting $c = \partial\langle s\rangle - \langle t\rangle$ so that $\partial c = -\partial\langle t\rangle$,

$$\int_{\langle t\rangle} \omega - \int_{\partial\langle t\rangle} \mu = \int_{\langle t\rangle} \omega + \int_{\partial c} \mu = \int_{\langle t\rangle} \omega + \int_c d\mu = \int_{\langle t\rangle} \omega + \int_c \omega = \int_{\partial\langle s\rangle} \omega = 0 \quad \text{(by hypothesis)}.$$

Applying (b_r), there exists a form μ' defined near $[t]$ such that $\mu' = \mu$ near near $[t^{k-2}]$ and $d\mu' = \omega$ near $[t]$. Let μ_2 be the form defined near $[s^{k-1}]$ by glueing together μ and μ' a1ong their common domain, an open set where they agree (Figure 6.8). Then $d\mu_2 = \omega$ near $[s^{k-1}]$, since both μ and μ' have this property in their domains of definition. .

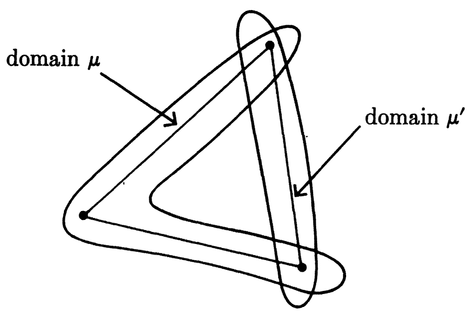


Figure 6.8

If $k=1$, then $[s^{k-1}]$ consists of two vertices v_0, v_1. Since ω is closed, Poincaré's Lemma guarantees the existence of smooth $(r-1)$-forms μ_i near v_i $(i=0,1)$, with $d\mu_i=\omega$. By shrinking domains, we can assume (domain μ_0) and (domain μ_1) are disjoint (Figure 6.9). This defines μ_2 near $[s^{k-1}]$, with $d\mu_2=\omega$ near $[s^{k-1}]$ as before.

Figure 6.9

Finally, let f be a smooth function which is identically 1 in a small neighborhood of $[s^{k-1}]$, and identically zero outside the domain of μ_2. Then $f\mu_2$ is a smooth $(r-1)$-form defined near $[s]$. Let $\tau=d(f\mu_2)$. Then τ is a closed r-form defined near $[s]$, and we have near $[s^{k-1}]$

$$\tau=d(f\mu_2)=df\wedge\mu_2+f\,d\mu_2=d\mu_2=\omega,$$

since $f\equiv 1$ and $df\equiv 0$ near $[s^{k-1}]$.

Proof of Lemma 2 We shall construct inductively a sequence

$$\tau_0,\,\tau_1,\ldots,\,\tau_n=\tau \quad (n=\dim X)$$

of $(\ell-1)$-forms such that

(1) τ_k is defined in a neighborhood of the k-skeleton $\left[K^k\right]$ of K,

(2) $d\tau_k=\omega$ near $\left[K^k\right]$,

(3) $\tau_k=\tau_{k-1}$ near $\left[K^{k-1}\right]$, and

(4) $\displaystyle\int_{\ell-1}(\tau_{\ell-1})=c.$

Note that this will prove the Lemma because (4) implies that for each oriented $(\ell-1)$-simplex $\langle s\rangle$ of $[K]$ and each $k\geq \ell-1$,

$$\int_{\ell-1}(\tau_k)(\langle s\rangle)=\int_{\langle s\rangle}\tau_k=\int_{\langle s\rangle}\tau_{\ell-1}=\int_{\ell-1}(\tau_{\ell-1})(\langle s\rangle)=c(\langle s\rangle).$$

To construct τ_0, cover $\left[K^0\right]$ by a collection of mutually disjoint balls. Since ω is closed, ω is exact in each of these balls by Poincaré's Lemma. Hence there exists a smooth $(\ell-1)$-form τ_0', defined on the union of these balls, such that $d\tau_0'=\omega$ there. If $\ell-1\neq 0$, take $\tau_0=\tau_0'$. If $\ell-1=0$, we want $\displaystyle\int_0(\tau_0)=c$. But for v_j a vertex of $[K]$,

$$\int_0(\tau_0')(\langle v_j\rangle)=\int_{\langle v_j\rangle}\tau_0'=\tau_0'(v_j).$$

Let $a_j=c(v_j)-\tau_0'(v_j)$, and define τ_0 on the ball about v_j by $frc\tau_0=\tau_0'+a_j$. Then $d\tau_0=d\tau_0'=\omega$ near $\left[K^0\right]$, and $\displaystyle\int_0(\tau_0)=c$ as required

Now assume that τ_{k-1} has been constructed with Properties $(1)-(4)$. To construct τ_k, note that if we can find, for each k-simplex s, a smooth $(\ell-1)$-form $\tau_k(s)$ defined in a neighborhood of $[s]$ such that $d\left(\tau_k(s)\right)=\omega$ near $[s]$, and $\tau_k(s)=\tau_{k-1}$ near $[s^{k-1}]$, then glueing will yield a smooth $(\ell-1)$-form τ_k' satisfying $(1)-(3)$.

To construct $\tau_k(s)$, we shall apply (b_ℓ) of Lemma 3. Note that ω is a smooth closed ℓ-form defined near $[s]$ and that τ_{k-1} is a smooth $(\ell-1)$-form defined near $[s^{k-1}]$ such that $d\tau_{k-1}=\omega$ near $[s^{k-1}]$. Furthermore, if $k=\ell$, then

$$\begin{aligned}
\int_{\langle s\rangle}\omega &= \int_{\ell}(\omega)\left(\langle s\rangle\right) \quad (\langle s\rangle = s \text{ together with either orientation})\\
&= \partial^* c\left(\langle s\rangle\right) \quad (\text{by hypothesis of Lemma 2})\\
&= c\left(\partial\langle s\rangle\right)\\
&= \int_{k-1}(\tau_{k-1})\left(\partial\langle s\rangle\right) \quad (\text{by (4) since } k=\ell)\\
&= \int_{\partial\langle s\rangle}\tau_{k-1}
\end{aligned}$$

Hence we can apply (b_ℓ), There exists a smooth $(\ell-1)$-form $\tau_k(s)$ near $[s]$ such that $\tau_k(s)=\tau_{k-1}$ near $[s^{k-1}]$ and $d\left(\tau_k(s)\right)=\omega$ near $[s]$.

This constructs τ_k' satisfying $(1)-(3)$. If $k\neq\ell-1$, set $\tau_k=\tau_k'$. If $k=\ell-1$, we have $\tau_{\ell-1}'$ satisfying $(1)-(3)$, and we want $\tau_{\ell-1}$ such that $\int_{\ell-1}(\tau_{\ell-1})=c$. Let $c_1=c-\int_{\ell-1}\left(\tau_{\ell-1}'\right)$, and define $\tau_{\ell-1}$ in a neighborhood of $\left[K^{\ell-1}\right]$ by

$$\tau_{\ell-1}=\tau_{\ell-1}'+\alpha_{\ell-1}(c_1),$$

where $\alpha_{\ell-1}$ is the linear map $C^{\ell-1}(K)\to C^\infty(X,\,\Lambda^{\ell-1}(X))$ defined in Lemma 1.

For each r and each oriented r-simplex $\langle s\rangle$, note that $\alpha_r\left(\varphi_{\langle s\rangle}\right)$ is identically zero on a neighborhood of $X\setminus \mathrm{St}\ (s)$. In particular, $\alpha_r\left(\varphi_{\langle s\rangle}\right)$ is identically zero near $\left[K^{r-1}\right]$. Since each $c\in C^r(K)$ is a linear combination of such $\varphi_{\langle s\rangle}$, $\alpha_r(c)$ is identically zero near $\left[K^{r-1}\right]$ for each r-cochain c.

Applying this first with $r=\ell$, then with $r=\ell-1$, we find

$$d\tau_{\ell-1}=d\tau_{\ell-1}'+d\circ\alpha_{\ell-1}(c_1)=d\tau_{\ell-1}'+\alpha_\ell\circ\partial^*(c_1)=d\tau_{\ell-1}'=\omega$$

near $\left[K^{r-1}\right]$ and

$$\tau_{\ell-1}=\tau_{\ell-1}'+\alpha_{\ell-1}(c_1)=\tau_{\ell-1}'=\tau_{\ell-2}$$

near $\left[K^{r-2}\right]$. Thus $\tau_{\ell-1}$ satisfies $(1)-(3)$ with $k=\ell-1$. Property (4) is also satisfied:

$$\int_{\ell-1}(\tau_{\ell-1})=\int_{\ell-1}\left(\tau_{\ell-1}'\right)+\int_{\ell-1}\circ\,\alpha_{\ell-1}(c_1)=(c-c_1)+c_1=c.$$

Remark 1. De Rham's theorem shows that the simplicial cohomology groups (with coefficients in $\mathbb{R}$) of a smoothly triangulated manifold $(X,\,K,\,h)$ are isomorphic to the De Rham cohomology groups of X. In particular, these groups are independent ofthe triangulation $(K,\,h)$ of X. Since the cohomology groups are dual to the homology groups, the groups $H_\ell(K,\,\mathbb{R})$, for $[K]$ a smooth manifold, also depend on $[K]$ only, not on the particular simplicial subdivision K.

Remark 2. The direct sum $\sum_{\ell=0}^{\dim X} \bigoplus H^\ell(X,\, d)$ can be given the structure of an associative algebra as follows. Recall that $\sum_{\ell=0}^{\dim X} \bigoplus C^\infty(X,\, \Lambda^\ell(X))$ is an associative algebra under exterior multiplication $\wedge$. $\sum_{\ell=0}^{\dim X} \bigoplus Z^\ell(X,\, d)$ is a subalgebra, for if $d\omega = 0$ and $d\tau = 0$, then

$$d\,(\omega \wedge \tau) = (d\omega) \wedge \tau \pm \omega \wedge (d\tau) = 0.$$

$\sum_{\ell=0}^{\dim X} \bigoplus B^\ell(X,\, d)$ is an ideal in $\sum_{\ell=0}^{\dim X} \bigoplus Z^\ell(X,\, d)$, for if $\omega = d\mu$ and $d\tau = 0$, then $\omega \wedge \tau = d\,(\mu \wedge \tau)$. Hence

$$\sum_{\ell=0}^{\dim X} \bigoplus H^\ell(X,\, d) = \sum_{\ell=0}^{\dim X} \bigoplus \left(Z^\ell(X,\, d)/B^\ell(X,\, d)\right) \cong \sum_{\ell=0}^{\dim X} \bigoplus \left(Z^\ell(X,\, d)/\sum_{\ell=0}^{\dim X} \bigoplus B^\ell(X,\, d)\right)$$

is also an associative algebra. In particular, $\sum_{\ell=0}^{\dim X} \bigoplus H^\ell(X,\, d)$ is a ring, called the *De Rham cohomology ring* of X.

It is also possible to define a product, called the *cup product,* of simplicial cochains in such a way that $\sum_{\ell=0}^{\dim X} \bigoplus H^\ell(K)$ becomes an algebra. It can be shown that the isomorphism $\int : \sum_{\ell=0}^{\dim X} \bigoplus H^\ell(X,\, d) \to \sum_{\ell=0}^{\dim X} \bigoplus H^\ell(K)$ is then an algebra isomorphism.

Remark 3. Lemma 3 contains in disguise a proof that

$$H^\ell(S^n,\, d) = \begin{cases} 0 & \text{if } 0 < \ell < n \\ \mathbb{R} & \text{if } \ell = 0,\, n. \end{cases}$$

For if ω is a closed ℓ-form $(0 < \ell < n)$ defined on a neighborhood of the n-skeleton $[s^n]$ of an $(n+1)$-simplex s, then it was shown that ω extends to a closed (and hence exact) ℓ-form on $[s]$. This implies that $H^\ell(S^n,\, d) = 0$ for $0 < \ell < n$. It was shown for $\ell = n$ that any closed n-form ω, defined near $[s^n]$ such that $\int_{\partial\langle s\rangle} \omega$ is also exact. The map $Z^n(S^n,\, d) \to \mathbb{R}$ defined by $\omega \to \int_{\partial\langle s\rangle} \omega$ is then a homomorphism with kernel $B^n(S^n,\, d)$. Hence $H^n(S^n,\, d) = \mathbb{R}$. Also $H^0(S^n,\, d) = \mathbb{R}$ because S^n is connected.

We have tacitly assumed here that any closed ℓ-form ω on S^n can be extended to a closed ℓ-form defined in a neighborhood of S^n. $\Psi^*\omega$ is such an extension, where $\Psi : \mathbb{R}^{n+1} \setminus \{0\} \to S^n$ is radial projection.